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# Spontaneous electromagnetic fluctuations in unmagnetized plasmas.

## II. Relativistic form factors of aperiodic thermal modes

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General expressions for the electromagnetic fluctuation spectra in unmagnetized plasmas are derived using fully relativistic dispersion functions and form factors for the important class of isotropic plasma particle distribution functions including in particular relativistic Maxwellian distributions. In order to obtain fluctuation spectra valid in the entire complex frequency plane, the proper analytical continuations of the unmagnetized form factors and dispersion functions are presented. The results are illustrated for the important special case of isotropic Maxwellian particle distribution functions providing in particular the thermal fluctuations of aperiodic modes. No restriction to the plasma temperature value is made, and the electromagnetic fluctuation spectra of ultrarelativistic thermal plasmas are calculated. The fully relativistic calculations also provide more general results in the limit of nonrelativistic plasma temperatures being valid in the entire complex frequency plane. They complement our earlier results in paper I and III of this series for negative values of the imaginary part of the frequency. A new collective, transverse, damped aperiodic mode with the damping rate  $\gamma \propto -k^{-5/3}$  is discovered in an isotropic thermal electron-proton plasma with nonrelativistic temperatures. © 2013 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4804402>]

## I. INTRODUCTION

Aperiodic fluctuations with  $\omega_R = 0$  are the extreme limiting case of weakly propagating fluctuations with frequencies  $\omega = \omega_R + i\gamma$  for which the real part of the frequency  $\omega_R \ll \gamma$  is much smaller than the imaginary part  $\gamma$ . The existence of collective (with a fixed frequency-wavenumber relation  $\gamma = \gamma(k)$ ) aperiodic modes in unmagnetized plasmas with anisotropic momentum distribution functions was analytically proven in 1959 by Weibel<sup>1</sup> and Fried.<sup>2</sup> Modern particle-in-cell plasma simulations<sup>3–7</sup> have confirmed these analytical results. The past research on collective aperiodic modes has recently been reviewed.<sup>8,9</sup>

The importance of aperiodic modes for astrophysical particle acceleration near cosmic shock waves was noted first by Medvedev and Loeb<sup>10</sup> pointing out that the underlying two-stream magnetic instability is capable of producing strong magnetic field turbulence in the internal and external shocks of gamma-ray burst sources. Such microturbulence is essential for gas flow entropisation to form shock wave structures,<sup>11</sup> because practically all shocks observed in outer space are collisionless. In such shocks, the thermalization of the upstream flow occurs via scattering of particles on the electromagnetic fluctuations that are self-consistently generated in multistream flows within the shock structure. The magnetic fluctuations therefore play a crucial role in the

formation of collisionless shocks both in magnetized and unmagnetized cosmic media. The low magnetic Mach number shocks, like the Earth bow shock, are formed at the scale of the ion Larmor radius. But most of astrophysical shocks have high Mach numbers so that the upstream field is small. The conventional picture emerged that when two oppositely directed plasma streams collide, the Weibel instability generates small-scale magnetic fields even in an initially nonmagnetized plasma. Charged particle scattering off these magnetic fluctuations provides an isotropization mechanism necessary for the shock transition to form. Astrophysical shocks are thus formed due to the generation of magnetic field turbulence. In order to calculate the formation time for unmagnetized collisionless shock waves<sup>12</sup> resulting from colliding plasma shells, the quantitative knowledge of the spontaneously emitted electromagnetic fluctuations in unmagnetized plasmas is important. The spontaneously emitted electromagnetic fluctuations define the initial stage of the turbulent fields near shock waves.

The Weibel instability in unmagnetized plasmas has also been proposed as a mechanism to generate cosmological seed magnetic fields.<sup>11,13–16</sup> This instability, which occurs in anisotropic bi-Maxwellian plasmas as well as counterstreaming plasmas, is essential for creating seed magnetic fields for any dynamo amplification process.

The generation of collective aperiodic fluctuations by plasma instabilities obviously requires the presence of velocity-anisotropic plasma particle distribution functions but cannot be driven in isotropic particle distribution

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functions.<sup>17</sup> However, Yoon<sup>18</sup> pointed out that non-collective aperiodic fluctuations are generated by the spontaneous emission of magnetic field fluctuations even in isotropic plasmas. This spontaneous emission is necessary to provide the seed perturbation for any instability. To drive an instability of a particular eigenmode in a plasma, one needs to have a spontaneous emission of that particular mode as a seed perturbation, so that the eigenmode can be amplified when the free energy (e.g., in the form of bi-Maxwellian or counterstream distributions) becomes available. In the first paper of this series<sup>19</sup>—hereafter referred to as paper I—general expressions for the electromagnetic fluctuation spectra from uncorrelated plasma particles in unmagnetized plasmas have been derived, which are covariantly correct within the theory of special relativity. The general expressions hold for unmagnetized plasmas of arbitrary composition and arbitrary momentum dependences of the plasma particle distribution functions and for collective and non-collective fluctuations. They are uniquely given in terms of the longitudinal and transverse plasma dispersion functions and the parallel and perpendicular form factors. For the important class of gyro-tropic plasma distribution functions, the general forms of the dispersion functions and form factors simplify.

In paper I and paper III<sup>20</sup> of this series, the results were illustrated for the important special cases of isotropic Maxwellian and generalized Kappa distribution functions, respectively, using *nonrelativistic* form factors and plasma dispersion functions. In particular, the calculated thermal fluctuations of aperiodic magnetic modes were then used<sup>21</sup> to determine the total strength of aperiodic turbulent magnetic field fluctuations in the unmagnetized intergalactic medium immediately after the reionization onset. These guaranteed magnetic fields with a high volume filling factor in the form of randomly distributed fluctuations, produced by the spontaneous emission of the isotropic thermal intergalactic medium plasma, then serve as seed fields for possible amplification by later possible plasma instabilities from anisotropic plasma particle distribution functions, magnetohydrodynamic instabilities, and/or the magnetohydrodynamic dynamo process. This mechanism of spontaneously emitted aperiodic turbulent seed magnetic fields should also operate during earlier cosmological epochs before recombination. This requires an extension of the calculations of the electromagnetic fluctuation spectra to relativistic plasma temperatures in excess of  $10^9$  K for electrons, which is reached at redshifts  $z > 10^5$ .

It is the purpose of the present investigation to derive the electromagnetic fluctuations using fully *relativistic* dispersion functions and form factors for the important class of isotropic plasma particle distribution functions including in particular relativistic Maxwellian distributions. In paper I, we were primarily interested in aperiodic fluctuations with positive values of  $\gamma > 0$ . As we demonstrate, the fully relativistic calculations agree favorably well with our earlier results<sup>19</sup> for  $\gamma > 0$  but also provide the correct behavior at negative values of  $\gamma$ , when the nonrelativistic temperature limit is taken. Moreover, we consider the ultrarelativistic limit of the thermal fluctuation spectra that are of interest for the unmagnetized fully ionized early Universe plasma and for the interior of hot and compact stars.

## II. ELECTROMAGNETIC FLUCTUATION SPECTRA

Using the system of the Klimontovich-Maxwell equations for the phase space density and electromagnetic field perturbations, we derived in paper I the electromagnetic fluctuation spectra (notation identical to paper I)

$$\begin{pmatrix} \langle \delta E_{\parallel}^2 \rangle_{k,\omega} \\ \langle \delta E_{\perp}^2 \rangle_{k,\omega} \\ \langle \delta B_{\perp}^2 \rangle_{k,\omega} \end{pmatrix} = \sum_a \frac{\omega_{p,a}^2 m_a}{4\pi^3 k^2} \begin{pmatrix} |K_{\parallel}(k, \omega)| \\ |\omega \Lambda_L(\vec{k}, \omega)|^2 \\ |K_{\perp}(k, \omega)| \\ |\omega \Lambda_T(\vec{k}, \omega)|^2 \\ c^2 k^2 |K_{\perp}(k, \omega)| \\ |\omega^2 \Lambda_T(\vec{k}, \omega)|^2 \end{pmatrix}, \quad (1)$$

where we correct for a missing  $|\omega|^{-2}$  in the expression for  $\langle \delta E_{\parallel}^2 \rangle_{k,\omega}$  and for the absolute values of the form factors  $K_{\parallel}(k, \omega)$  and  $K_{\perp}(k, \omega)$ . Coulomb's law then provides for the charge density fluctuations

$$\begin{aligned} \langle \delta \rho^2 \rangle_{k,\omega} &= \frac{k^2 \langle \delta E_{\parallel}^2 \rangle_{k,\omega}}{(4\pi)^2} \\ &= \sum_a \frac{\omega_{p,a}^2 m_a}{2(2\pi)^5} \frac{|K_{\parallel}(k, \omega)|}{|\omega \Lambda_L(\vec{k}, \omega)|^2}, \end{aligned} \quad (2)$$

and the continuity equation  $\dot{\rho} + \text{div} \vec{j} = 0$  yields for the parallel current density fluctuations

$$\begin{aligned} \langle \delta J_{\parallel}^2 \rangle_{k,\omega} &= \frac{|\omega|^2 \langle \delta \rho^2 \rangle_{k,\omega}}{k^2} \\ &= \sum_a \frac{\omega_{p,a}^2 m_a}{2(2\pi)^5 k^2} \frac{|K_{\parallel}(k, \omega)|}{|\Lambda_L(\vec{k}, \omega)|^2}. \end{aligned} \quad (3)$$

Ampere's law gives the perpendicular current density fluctuations

$$\begin{aligned} \langle \delta J_{\perp}^2 \rangle_{k,\omega} &= \frac{|\omega|^2}{(4\pi)^2} \left[ 1 + \frac{c^4 k^4}{|\omega|^4} \right] \langle \delta E_{\perp}^2 \rangle_{k,\omega} \\ &= \sum_a \frac{\omega_{p,a}^2 m_a}{2(2\pi)^5 k^2} \left[ 1 + \frac{c^4 k^4}{|\omega|^4} \right] \frac{|K_{\perp}(k, \omega)|}{|\Lambda_T(\vec{k}, \omega)|^2}. \end{aligned} \quad (4)$$

Eqs. (2) and (4) correct the erroneous expressions (23) in paper I.

As before the fluctuation spectra depend on longitudinal ( $\Lambda_L(\vec{k}, \omega)$ ) and transverse ( $\Lambda_T(\vec{k}, \omega)$ ) dispersion functions and the form factors

$$\begin{aligned} \begin{pmatrix} K_{\parallel}^+(k, \omega) \\ K_{\perp}^+(k, \omega) \end{pmatrix} &= k^2 \Re \left[ i \int d^3 p \frac{f_a(\vec{p})}{\omega - kv_{\parallel}} \begin{pmatrix} v_{\parallel}^2 \\ v_{\perp}^2 \end{pmatrix} \right] \\ &= k^2 \Re \left[ -i \int d^3 p \frac{f_a(\vec{p})}{kv_{\parallel} - (r + i\gamma)} \begin{pmatrix} v_{\parallel}^2 \\ v_{\perp}^2 \end{pmatrix} \right], \end{aligned} \quad (5)$$

with the complex frequency  $\omega = r + i\gamma$ , and where without loss of generality we orient the fluctuation wave vector  $\vec{k} = (0, 0, k)$  along the  $z$ -axis. The index  $+$  indicates that the expressions hold for positive imaginary frequencies  $\gamma > 0$ . For negative imaginary frequencies  $\gamma < 0$ , proper analytical continuation is required.

## A. Relativistic gyrotropic distribution functions

For gyrotropic particle distribution functions

$$f_a(p_{\parallel}, p_{\perp}, \phi) = \frac{1}{2\pi} f_a(p_{\parallel}, p_{\perp}), \quad (6)$$

the form factors (5) reduce to

$$\begin{aligned} \left( \begin{array}{c} K_{\parallel}^{+}(k, \omega) \\ K_{\perp}^{+}(k, \omega) \end{array} \right) &= k^2 \Re \left[ -i \int_{-\infty}^{\infty} dp_{\parallel} \right. \\ &\quad \left. \times \int_0^{\infty} dp_{\perp} p_{\perp} \frac{f_a(p_{\parallel}, p_{\perp})}{kv_{\parallel} - \omega} \left( \frac{v_{\parallel}^2}{v_{\perp}^2} \right) \right]. \end{aligned} \quad (7)$$

These two form factors are symmetric in the wavenumber value  $K_{\parallel}^{\pm}(-k, \omega) = K_{\parallel}^{\pm}(k, \omega)$ , so that without loss of generality we may restrict the following analysis to positive values of  $k$ .

For the further evaluation of the form factors, the well-known complication arises that in a relativistic treatment  $v_{\parallel} = p_{\parallel}/m_a\Gamma$  depends through the Lorentz factor  $\Gamma = \sqrt{m_a^2c^2 + p_{\perp}^2 + p_{\parallel}^2}/m_ac$  also on the perpendicular momentum  $p_{\perp}$ , so that the momentum integration over  $p_{\parallel}$  and  $p_{\perp}$  in Eq. (5) no longer decouples as in the nonrelativistic analysis. To remedy this complication, one transforms to two new variables of integration<sup>22</sup>

$$y = p_{\parallel}/m_ac, \quad E = \Gamma = \sqrt{1 + \frac{p_{\parallel}^2 + p_{\perp}^2}{m_a^2c^2}}. \quad (8)$$

The form factors (7) then read

$$K_{\parallel}^{+}(k, z) = kc(m_ac)^3 \Re \left[ -i \int_1^{\infty} dE \int_{-\sqrt{E^2-1}}^{\sqrt{E^2-1}} dy \frac{y^2 f_a(E, y)}{y - Ez} \right] \quad (9)$$

and

$$\begin{aligned} K_{\perp}^{+}(k, z) &= kc(m_ac)^3 \Re \left[ -i \int_1^{\infty} dE \right. \\ &\quad \left. \times \int_{-\sqrt{E^2-1}}^{\sqrt{E^2-1}} dy \frac{(E^2 - 1 - y^2) f_a(E, y)}{y - Ez} \right], \end{aligned} \quad (10)$$

where we introduce the complex phase speed

$$z = \frac{\omega}{kc} = R + iI, \quad I = \frac{\gamma}{kc}, \quad R = \frac{\omega_R}{kc}. \quad (11)$$

Correspondingly, the longitudinal and transverse dispersion functions for gyrotropic plasma distribution functions are given by<sup>17</sup>

$$\begin{aligned} \Lambda_L^{+}(k, z) &= 1 - \sum_a \frac{\omega_{p,a}^2 (m_ac)^3}{k^2 c^2 z} \int_1^{\infty} dE E \\ &\quad \times \int_{-\sqrt{E^2-1}}^{\sqrt{E^2-1}} dy \frac{y}{y - Ez} \left[ \frac{\partial f_a}{\partial y} + \frac{y}{E} \frac{\partial f_a}{\partial E} \right] \end{aligned} \quad (12)$$

and

$$\begin{aligned} \Lambda_T^{+}(k, z) &= 1 - \frac{1}{z^2} - \sum_a \frac{\omega_{p,a}^2 (m_ac)^3}{2k^2 c^2 z^2} \int_1^{\infty} dE \\ &\quad \times \int_{-\sqrt{E^2-1}}^{\sqrt{E^2-1}} dy \frac{E^2 - 1 - y^2}{y - Ez} \left[ \frac{\partial f_a}{\partial y} + z \frac{\partial f_a}{\partial E} \right], \end{aligned} \quad (13)$$

respectively.

Throughout this work, we investigate exclusively isotropic, gyrotropic distribution functions  $f_a(E)$ . For such distribution functions, the longitudinal and transverse dispersion functions (12) and (13) also are symmetric in  $k$ , justifying the restriction of the following analysis to positive values of  $k$ .

## B. Isotropic distribution functions

Isotropic (and gyrotropic) plasma particle distribution functions are given by

$$f_a(p_{\parallel}, p_{\perp}) = N_a f_a(E), \quad (14)$$

where the normalization constant  $N_a$  is determined by the requirement  $\int d^2p f_a(p_{\parallel}, p_{\perp}) = 1$ , providing

$$N_a = \left[ 2(m_ac)^3 \int_1^{\infty} dE E \sqrt{E^2 - 1} f_a(E) \right]^{-1}. \quad (15)$$

In particular, we will consider isotropic thermal Maxwell-Jüttner distribution functions

$$f_a(E) = e^{-\mu_a E}, \quad \mu_a = \frac{m_a c^2}{k_B T}, \quad (16)$$

with the normalization factor

$$N_a = \frac{\mu_a}{2(m_ac)^3 K_2(\mu_a)}, \quad (17)$$

which obey

$$\frac{\partial f_a}{\partial E} = -\mu_a e^{-\mu_a E} = -\mu_a f_a(E). \quad (18)$$

## III. DISPERSION FUNCTIONS AND FORM FACTORS FOR ISOTROPIC RELATIVISTIC DISTRIBUTION FUNCTIONS

Besides the derivation of the dispersion functions and form factors for general isotropic distribution functions, we will also consider the special case of thermal distributions. We start with the latter.

### A. Thermal distributions

For thermal distributions (16), we introduce the functions

$$L(z, \mu_a) = \frac{\mu_a}{2K_2(\mu_a)z} \int_1^{\infty} dE e^{-\mu_a E} \int_{-\sqrt{E^2-1}}^{\sqrt{E^2-1}} dy \frac{y^2}{y - Ez} \quad (19)$$



and

$$T(z, \mu_a) = \frac{\mu_a}{2K_2(\mu_a)z} \int_1^\infty dE e^{-\mu_a E} \times \int_{-\sqrt{E^2-1}}^{\sqrt{E^2-1}} dy \frac{E^2 - 1 - y^2}{y - Ez}. \quad (20)$$

Using the property (18) and the normalization factor (17), the longitudinal form factor (9) and the longitudinal dispersion function (12) are then given by

$$K_{\parallel}^+(k, z) = kc \Re[-izL(z, \mu_a)] \quad (21)$$

and

$$\Lambda_L^+(k, z) = 1 + \sum_a \frac{\omega_{p,a}^2 \mu_a}{k^2 c^2} L(z, \mu_a). \quad (22)$$

Likewise, the transverse form factor (10) and the transverse dispersion function (13) read

$$K_{\perp}^+(k, z) = kc \Re[-izT(z, \mu_a)] \quad (23)$$

and

$$\Lambda_T^+(k, z) = 1 - \frac{1}{z^2} + \sum_a \frac{\omega_{p,a}^2 \mu_a}{2k^2 c^2} T(z, \mu_a). \quad (24)$$

Using

$$\begin{aligned} \int_{-\sqrt{E^2-1}}^{\sqrt{E^2-1}} dy \frac{y^2}{y - Ez} &= \int_{-\sqrt{E^2-1}}^{\sqrt{E^2-1}} dy \frac{y^2 - E^2 z^2 + E^2 z^2}{y - Ez} \\ &= \int_{-\sqrt{E^2-1}}^{\sqrt{E^2-1}} dy (y + Ez) - E^2 z^2 J^+(E, z) \\ &= 2\sqrt{E^2 - 1} Ez - E^2 z^2 J^+(E, z), \end{aligned} \quad (25)$$

in terms of the integral

$$J^+(E, z) = - \int_{-\sqrt{1-E^2}}^{\sqrt{1-E^2}} \frac{dt}{t - z}, \quad (26)$$

the function (19) can be further reduced, using Eqs. (15) and (17), to

$$L(z, \mu_a) = 1 - \frac{\mu_a z}{2K_2(\mu_a)} \int_1^\infty dE E^2 e^{-\mu_a E} J^+(E, z). \quad (27)$$

Likewise, using

$$\begin{aligned} \int_{-\sqrt{E^2-1}}^{\sqrt{E^2-1}} dy \frac{E^2 - 1 - y^2}{y - Ez} &= \int_{-\sqrt{E^2-1}}^{\sqrt{E^2-1}} dy \frac{E^2 - 1 - E^2 z^2 - (y^2 - E^2 z^2)}{y - Ez} \\ &= -[E^2(1 - z^2) - 1]J^+(E, z) - 2\sqrt{E^2 - 1}Ez, \end{aligned} \quad (28)$$

the function (20) becomes

$$T(z, \mu_a) = -1 - \frac{\mu_a}{2K_2(\mu_a)z} \int_1^\infty dE e^{-\mu_a E} \times [E^2(1 - z^2) - 1]J^+(E, z). \quad (29)$$

Introducing the moments

$$\begin{aligned} w_2(\mu_a, E) &= \int_E^\infty dx x^2 e^{-\mu_a x} = \frac{E^2 e^{-\mu_a E}}{\mu_a} M(\mu_a, E), \\ w_0(\mu_a, E) &= \int_E^\infty dx e^{-\mu_a x} = \frac{e^{-\mu_a E}}{\mu_a}, \\ M(\mu_a, E) &= 1 + \frac{2}{\mu_a E} + \frac{2}{\mu_a^2 E^2}, \end{aligned} \quad (30)$$

we obtain for the functions (27) and (29)

$$L(z, \mu_a) = 1 + \frac{\mu_a z}{2K_2(\mu_a)} \int_1^\infty dE \frac{\partial w_2(\mu_a, E)}{\partial E} J^+(E, z) \quad (31)$$

and

$$\begin{aligned} T(z, \mu_a) &= -1 + \frac{\mu_a}{2K_2(\mu_a)z} \\ &\times \int_1^\infty dE \frac{\partial}{\partial E} [(1 - z^2)w_2(E, \mu_a) - w_0(E, \mu_a)] J^+(E, z). \end{aligned} \quad (32)$$

In order to derive form factors and dispersion functions valid in the entire complex  $I$ -plane, it is necessary to analytically continue the function  $J^+(E, z)$  to negative values of  $I = \Im(z) < 0$ , which is detailed in Appendix A.

## B. General isotropic distribution functions

For general isotropic distribution (14), the plasma dispersion functions (12) and (13) simplify to

$$\begin{aligned} \Lambda_L^+(k, z) &= 1 - \sum_a \frac{N_a \omega_{p,a}^2 (m_a c)^3}{k^2 c^2 z} \int_1^\infty dE \frac{\partial f_a(E)}{\partial E} \\ &\times \int_{-\sqrt{E^2-1}}^{\sqrt{E^2-1}} dy \frac{y^2}{y - Ez} \\ &= 1 - \sum_a \frac{2N_a \omega_{p,a}^2 (m_a c)^3}{k^2 c^2} \left( \int_1^\infty dE E \sqrt{E^2 - 1} \frac{\partial f_a(E)}{\partial E} \right. \\ &\quad \left. - \frac{z}{2} \int_1^\infty dE E^2 \frac{\partial f_a(E)}{\partial E} J^+(E, z) \right) \end{aligned} \quad (33)$$

$$\begin{aligned} \Lambda_T^+(k, z) &= 1 - \frac{1}{z^2} \\ &+ \sum_a \frac{N_a \omega_{p,a}^2 (m_a c)^3}{k^2 c^2} \int_1^\infty dE E \sqrt{E^2 - 1} \frac{\partial f_a(E)}{\partial E} \\ &+ \sum_a \frac{N_a \omega_{p,a}^2 (m_a c)^3}{2k^2 c^2 z} \int_1^\infty dE \frac{\partial f_a(E)}{\partial E} \\ &\times [E^2(1 - z^2) - 1]J^+(E, z), \end{aligned} \quad (34)$$

in terms of the integral (26).

Likewise, the form factors (9) and (10) can be expressed with the same integral (26) as

$$K_{\parallel}^{+}(k, z) = kc\Re[-iz] + kc(m_ac)^3 N_a \int_1^{\infty} dE E^2 f_a(E) \Re[iz^2 J^{+}(E, z)] \quad (35)$$

and

$$K_{\perp}^{+}(k, z) = kc\Re[iz] + kc(m_ac)^3 N_a \int_1^{\infty} dE f_a(E) \times \Re\left[i[E^2(1-z^2) - 1]J^{+}(E, z)\right], \quad (36)$$

where in the first terms on the right-hand side of Eqs. (35) and (36), we used the normalization condition (15).

Now, it is convenient to introduce the positive functions  $u_n(E)$  for  $n = 0, 1, 2$  by

$$\frac{\partial u_n}{\partial E} = E^n \frac{\partial f_a}{\partial E}, \quad (37)$$

implying

$$\frac{\partial}{\partial E}[u_n - E^n f_a] = -nE^{n-1}f_a,$$

so that

$$u_n = E^n f_a(E) + n \int_E^{\infty} dx x^{n-1} f_a(x) > 0, \quad (38)$$

because  $\lim_{E \rightarrow \infty} E^2 f_a(E) \rightarrow 0$ . We then obtain for the dispersion relations (33) and (34)

$$\Lambda_L^{+}(k, z) = 1 - \sum_a \frac{2N_a \omega_{p,a}^2 (m_ac)^3}{k^2 c^2} \times \left( \int_1^{\infty} dE \sqrt{1-E^{-2}} \frac{\partial u_2(E)}{\partial E} - \frac{z}{2} \int_1^{\infty} dE \frac{\partial u_2(E)}{\partial E} J^{+}(E, z) \right) \quad (39)$$

and

$$\Lambda_T^{+}(k, z) = 1 - \frac{1}{z^2} + \sum_a \frac{N_a \omega_{p,a}^2 (m_ac)^3}{k^2 c^2} \int_1^{\infty} dE \sqrt{1-E^{-2}} \frac{\partial u_2(E)}{\partial E} + \sum_a \frac{N_a \omega_{p,a}^2 (m_ac)^3}{2k^2 c^2 z} \times \int_1^{\infty} dE J^{+}(E, z) \frac{\partial}{\partial E} \times [(1-z^2)u_2(E) - u_0(E)]. \quad (40)$$

For the evaluation of the form factors, we introduce the general moments

$$v_n(E) = \int_E^{\infty} dx x^n f_a(x) \quad (41)$$

to obtain

$$K_{\parallel}^{+}(k, z) = kc\Re[-iz] - kc(m_ac)^3 N_a \int_1^{\infty} dE \Re[iz^2 J^{+}(E, z)] \frac{\partial v_2(E)}{\partial E} \quad (42)$$

and

$$K_{\perp}^{+}(k, z) = kc\Re[iz] - kc(m_ac)^3 N_a \int_1^{\infty} dE \times \Re\left[iJ^{+}(E, z) \left( (1-z^2) \frac{\partial v_2(E)}{\partial E} - \frac{\partial v_0(E)}{\partial E} \right)\right]. \quad (43)$$

### C. Analytical continuation

The derivation of dispersion functions and form factors valid in the entire complex frequency (or phase speed) plane requires the proper analytical continuation of the function  $J(E, z)$ , which is detailed in Appendix A and provides

$$J(S, R, I) = \frac{1}{2} \ln \frac{(S+R)^2 + I^2}{(S-R)^2 + I^2} - i \left( \arctan \frac{S-R}{I} + \arctan \frac{S+R}{I} + \pi \sigma \Theta[1-R] \Theta[E-E_c(R)] \right), \quad (44)$$

with the plasma particle speed in units of the speed of light

$$S(E) = \sqrt{1-E^{-2}}, \quad (45)$$

and where  $\Theta[x]$  denotes the step function,  $\sigma = 0, 1, 2$  for  $I > 0, = 0, < 0$  and the characteristic Lorentz factor

$$E_c(R) = (1-R^2)^{-1/2}. \quad (46)$$

For aperiodic ( $R = 0$ ) fluctuations, we obtain

$$J(S, R = 0, I) = -2i \left[ \arctan \frac{S}{I} + \frac{\pi \sigma}{2} \right]. \quad (47)$$

Consequently, for any distribution function, the correct analytical continuations for the dispersion functions and form factors, valid in the entire complex frequency plane, result by replacing the function  $J^{+}(E, z)$  by  $J(E, z)$  in the expressions derived before. In the following sections, we analyze the special case of thermal aperiodic fluctuations in more detail.

### IV. THERMAL APERIODIC FLUCTUATIONS

During cosmological epochs before recombination at redshifts  $z \gg 1300$ , the unmagnetized baryonic intergalactic medium was a fully ionized electron-proton plasma<sup>23</sup> with the temperature  $k_B T_p = k_B T_e = k_B T \gg 13.6$  eV, so that especially for electrons relativistic corrections are important. We derive here the thermal aperiodic turbulent electromagnetic field fluctuation spectra.

According to Sec. III A, we have to calculate the two integrals (31) and (32) using the expression (47) for aperiodic fluctuations. We obtain with Eqs. (21) and (30)

$$\begin{aligned} \frac{K_{\parallel}(k, I)}{kcI} &= L(I, \mu_a) \\ &= 1 + \frac{\mu_a I}{K_2(\mu_a)} \int_1^{\infty} dE \frac{\partial w_2(\mu_a, E)}{\partial E} \left[ \arctan \frac{S}{I} + \frac{\pi\sigma}{2} \right] \\ &= 1 - \frac{\pi\sigma\mu_a I w_2(\mu_a, 1)}{2K_2(\mu_a)} \\ &\quad - \frac{I^2\mu_a}{K_2(\mu_a)} \int_1^{\infty} dE \frac{w_2(\mu_a, E)}{E^2\sqrt{E^2-1}[I^2+1-E^{-2}]} \\ &= 1 - \frac{\pi\sigma I}{2e^{\mu_a} K_2(\mu_a)} \left[ 1 + \frac{2}{\mu_a} + \frac{2}{\mu_a^2} \right] \\ &\quad - \frac{I^2}{K_2(\mu_a)} \int_1^{\infty} dE \frac{e^{-\mu_a E} M(\mu_a, E)}{\sqrt{E^2-1}[I^2+1-E^{-2}]} \end{aligned} \quad (48)$$

and

$$\begin{aligned} \frac{K_{\perp}(k, I)}{kcI} &= T(I, \mu_a) = -1 - \frac{\mu_a}{IK_2(\mu_a)} \int_1^{\infty} dE \left[ \arctan \frac{S}{I} + \frac{\pi\sigma}{2} \right] \\ &\quad \times \frac{\partial}{\partial E} [(1+I^2)w_2(\mu_a, E) - w_0(\mu_a, E)] \\ &= -1 + \frac{\pi\sigma\mu_a}{2IK_2(\mu_a)} [(1+I^2)w_2(\mu_a, E=1) \\ &\quad - w_0(\mu_a, E=1)] + \frac{\mu_a}{K_2(\mu_a)} \\ &\quad \times \int_1^{\infty} dE \frac{(1+I^2)w_2(\mu_a, E) - w_0(\mu_a, E)}{E^2\sqrt{E^2-1}[I^2+1-E^{-2}]} \\ &= -1 + \frac{\pi\sigma}{2Ie^{\mu_a} K_2(\mu_a)} \left[ (1+I^2) \left( 1 + \frac{2}{\mu_a} + \frac{2}{\mu_a^2} \right) - 1 \right] \\ &\quad + \frac{1}{K_2(\mu_a)} \int_1^{\infty} dE \frac{e^{-\mu_a E}}{\sqrt{E^2-1}[I^2+1-E^{-2}]} \\ &\quad \times \left[ (1+I^2)M(\mu_a, E) - \frac{1}{E^2} \right], \end{aligned} \quad (49)$$

where we partially integrated with respect to  $E$  and inserted the moments (30).

Substituting  $E = \cosh t$  yields

$$\begin{aligned} L(I, \mu_a) &= 1 - A_L(I, \mu_a) - \frac{I^2}{K_2(\mu_a)} \\ &\quad \times \int_0^{\infty} dt \frac{e^{-\mu_a \cosh t}}{I^2 + \tanh^2 t} \left[ 1 + \frac{2}{\mu_a \cosh t} + \frac{2}{\mu_a^2 \cosh^2 t} \right], \end{aligned} \quad (50)$$

$$\begin{aligned} T(I, \mu_a) &= -1 + A_T(I, \mu_a) + \frac{1}{K_2(\mu_a)} \int_0^{\infty} dt \frac{e^{-\mu_a \cosh t}}{I^2 + \tanh^2 t} \\ &\quad \times \left[ (1+I^2) \left( 1 + \frac{2}{\mu_a \cosh t} + \frac{2}{\mu_a^2 \cosh^2 t} \right) - \frac{1}{\cosh^2 t} \right], \end{aligned} \quad (51)$$

with the analytic continuations

$$\begin{aligned} A_T(I, \mu_a) &= \frac{\pi\sigma}{2Ie^{\mu_a} K_2(\mu_a)} \left[ (1+I^2) \left( 1 + \frac{2}{\mu_a} + \frac{2}{\mu_a^2} \right) - 1 \right], \\ A_L(I, \mu_a) &= \frac{\pi\sigma I}{2e^{\mu_a} K_2(\mu_a)} \left[ 1 + \frac{2}{\mu_a} + \frac{2}{\mu_a^2} \right]. \end{aligned} \quad (52)$$

Using  $\cosh^{-2} t = 1 + I^2 - (I^2 + \tanh^2 t)$  in Eqs. (50) and (51), we find with the integral representation of Bessel functions<sup>24</sup>

$$K_0(\mu_a) = \int_0^{\infty} dt e^{-\mu_a \cosh t}, \quad (53)$$

that

$$\begin{aligned} L(I, \mu_a) &= 1 - A_L(I, \mu_a) + \frac{2I^2 K_0(\mu_a)}{\mu_a^2 K_2(\mu_a)} - \frac{I^2}{K_2(\mu_a)} \\ &\quad \times \left( \left[ 1 + \frac{2(1+I^2)}{\mu_a^2} \right] Y(I, \mu_a) + \frac{2}{\mu_a} \int_{\mu_a}^{\infty} ds Y(I, s) \right) \end{aligned} \quad (54)$$

and

$$\begin{aligned} T(I, \mu_a) &= -1 + A_T(I, \mu_a) + \left[ 1 - \frac{2(1+I^2)}{\mu_a^2} \right] \frac{K_0(\mu_a)}{K_2(\mu_a)} \\ &\quad + \frac{2(1+I^2)}{\mu_a K_2(\mu_a)} \left[ \int_{\mu_a}^{\infty} ds Y(I, s) + \frac{1+I^2}{\mu_a} Y(I, \mu_a) \right] \end{aligned} \quad (55)$$

in terms of the single function

$$Y(I, \mu_a) = \int_0^{\infty} dt \frac{e^{-\mu_a \cosh t}}{I^2 + \tanh^2 t}, \quad (56)$$

and its first integral with respect to  $\mu_a$ .

Consequently, according to Eq. (1), the thermal aperiodic electromagnetic fluctuation spectra (recall that  $k$  stands for  $|k|$ )

$$\langle \delta E_{\parallel}^2 \rangle_{k, I} = \sum_a \frac{\omega_{p,a}^2 m_a}{4\pi^3 k^3 c |I|} \frac{|L(I, \mu_a)|}{\left| 1 + \sum_a \frac{\omega_{p,a}^2 \mu_a}{k^2 c^2} L(I, \mu_a) \right|^2} \quad (57)$$

and

$$\begin{aligned} \langle \delta B_{\perp}^2 \rangle_{k, I} &= \frac{\langle \delta E_{\perp}^2 \rangle_{k, I}}{I^2} = \sum_a \frac{\omega_{p,a}^2 m_a}{4\pi^3 k^3 c} \\ &\quad \times \frac{|IT(I, \mu_a)|}{\left| 1 + I^2 + \frac{1}{2} \sum_a \frac{\omega_{p,a}^2 \mu_a}{k^2 c^2} I^2 T(I, \mu_a) \right|^2} \end{aligned} \quad (58)$$

are determined by the two functions  $L(I, \mu_a)$  and  $T(I, \mu_a)$ , respectively. The functions  $L(I, \mu_a)$  and  $T(I, \mu_a)$  provide the full exact aperiodic thermal fluctuation spectra, holding for all imaginary frequency values  $\gamma \in [-\infty, \infty]$  and wavenumbers  $k \in [-\infty, \infty]$ . They generalize the earlier expressions

of paper I, derived from the nonrelativistic form factors and dispersion functions.

We next study the nonrelativistic ( $\mu_a \gg 1$ ) and ultrarelativistic ( $\mu_a \ll 1$ ) temperature limits.

### A. Nonrelativistic limit

Because of the large values  $\mu_a \gg 1$ , the main contribution to the integral (57) is provided by small values  $t \ll 1$ , so that we approximate

$$\tanh t \simeq t, \quad \cosh t = 1 + \frac{t^2}{2}, \quad (59)$$

yielding

$$Y(x) \simeq e^{-\mu_a} \int_0^\infty dt \frac{e^{-\frac{\mu_a t^2}{2}}}{t^2 + I^2} = \frac{\pi e^{-\mu_a}}{2|I|} D(x), \quad (60)$$

where we introduce

$$D(x) = e^{x^2} \operatorname{erfc}(x) = e^{x^2} \left[ 1 - \frac{2}{\pi^{1/2}} \int_0^x dt e^{-t^2} \right], \quad (61)$$

in terms of the complementary error function<sup>24</sup> and the variable

$$x = \sqrt{\frac{\mu_a I^2}{2}} = \sqrt{\frac{m_a \gamma^2}{2k_B T_a k^2}} = \left| \frac{\gamma}{k u_a} \right|, \quad (62)$$

where we have used the definitions (11) and (16) and introduced the nonrelativistic thermal velocity  $u_a = \sqrt{2k_B T_a / m_a}$ . The function (61) obeys

$$\frac{dD(x)}{dx} = 2xD(x) - \frac{2}{\sqrt{\pi}}. \quad (63)$$

For nonrelativistic temperatures  $\mu_a \gg 1$ , we use the integral representation of Bessel functions<sup>24</sup>

$$K_\nu(\mu_a) \simeq \sqrt{\frac{\pi}{2\mu_a}} e^{-\mu_a} \left[ 1 + \frac{4\nu^2 - 1}{8\mu_a} \right], \quad (64)$$

which readily provides

$$\frac{1}{K_2(\mu_a)} \simeq \sqrt{\frac{2\mu_a}{\pi}} e^{\mu_a}. \quad (65)$$

Consequently, with  $(2/\mu_a) = \beta_a^2 = U_a^2/c^2$  Eqs. (54) and (55) can be approximated to leading orders in  $\beta_a^2 \ll 1$  as

$$\begin{aligned} L(x) &\simeq 1 - \pi^{1/2} x \sigma \frac{I}{|I|} (1 + \beta_a^2) + \frac{\beta_a^6 x^2}{2} \\ &\quad - \pi^{1/2} x D(x) \left[ 1 + \frac{\beta_a^4}{2} (1 + \beta_a^2 x^2) \right] - \frac{\pi^{1/2} I^2}{x} e^{\mu_a} U(I, \mu_a) \end{aligned} \quad (66)$$

and

$$\begin{aligned} T(x) &\simeq -\beta_a^2 + \frac{\pi^{1/2} \sigma I}{x|I|} (1 + x^2 (1 + \beta_a^2)) \\ &\quad + \frac{\pi^{1/2} (1 + I^2)}{x} \left[ e^{\mu_a} U(I, \mu_a) + \frac{\beta_a^2}{2} (1 + I^2) D(x) \right] \end{aligned} \quad (67)$$

with

$$U(I, \mu_a) = \int_{\mu_a}^\infty ds e^{-s} D\left(\frac{|I|s^{1/2}}{2^{1/2}}\right). \quad (68)$$

With the property (63), we obtain for the last integral after partial integration

$$\begin{aligned} U(I, \mu_a) &= e^{-\mu_a} D(x) + \int_{\mu_a}^\infty ds e^{-s} \left[ \frac{I^2}{2} D\left(\frac{|I|s^{1/2}}{2^{1/2}}\right) - \frac{|I|}{\sqrt{2\pi s}} \right] \\ &= e^{-\mu_a} D(x) + \frac{I^2}{2} U(I, \mu_a) - \frac{|I|}{\sqrt{2}} \operatorname{erfc}(\mu_a^{1/2}), \end{aligned} \quad (69)$$

so that

$$\begin{aligned} G(x, \beta_a) &= e^{\mu_a} U(I, \mu_a) = e^{\mu_a} \int_{\mu_a}^\infty ds e^{-s} D\left(\frac{|I|s^{1/2}}{2^{1/2}}\right) \\ &= \frac{D(x) - \frac{|I|}{\sqrt{2}} D(\mu_a^{1/2})}{1 - \frac{I^2}{2}} = \frac{D(x) - \frac{\beta_a x}{\sqrt{2}} D\left(\frac{\sqrt{2}}{\beta_a}\right)}{1 - \frac{\beta_a^2 x^2}{2}}. \end{aligned} \quad (70)$$

In Fig. 1, we plot the deviation of ratio of the functions  $G(x, \beta_a)/D(x) - 1$  from unity, for three different nonrelativistic thermal velocities. As can be seen from Fig. 1, and as proven in Appendix B, the function  $G(x, \beta_a)$

$$G(x, \beta_a) \simeq D(x) \quad (71)$$

is basically identical to  $D(x)$  for all nonrelativistic thermal velocities. With this approximation, we find for the functions (66) and (67) with  $I^2 = \beta_a^2 x^2$

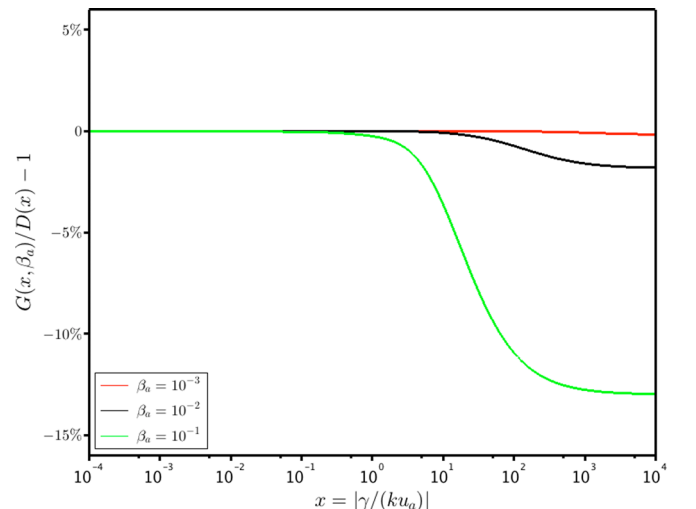


FIG. 1. Deviation in percentage of the ratio of the two functions  $G(x, \beta_a)/D(x)$  from unity calculated for three values of  $\beta_a = 10^{-3}, 10^{-2}$ , and  $\beta_a = 0.1$ .



$$L(x) \simeq 1 - \pi^{1/2}(1 + \beta_a^2)x[D(x) - \sigma] \quad (72)$$

and

$$T(x) \simeq \frac{\pi^{1/2}}{x} [(1 + \beta_a^2 x^2)D(x) - \sigma(1 + (1 + \beta_a^2)x^2)] - \beta_a^2. \quad (73)$$

The nonrelativistic limit of the thermal aperiodic fluctuation spectra (57) and (58) then is with  $\mu_a = 2c^2/u_a^2$

$$\begin{aligned} \langle \delta E_{\parallel}^2 \rangle_{k,x} &\simeq \sum_a \frac{\omega_{p,a}^2 m_a}{4\pi^3 k^3 u_a^2} \\ &\times \frac{|1 - \pi^{1/2}(1 + \beta_a^2)x[D(x) - \sigma]|}{\left|1 + 2 \sum_a \frac{\omega_{p,a}^2}{k^2 u_a^2} (1 - \pi^{1/2}(1 + \beta_a^2)x[D(x) - \sigma])\right|^2} \end{aligned} \quad (74)$$

and

$$\begin{aligned} \langle \delta B_{\perp}^2 \rangle_{k,x} &= \frac{\langle \delta E_{\perp}^2 \rangle_{k,x}}{\beta_a^2 x^2} = \sum_a \frac{\omega_{p,a}^2 m_a \beta_a}{4\pi^3 k^3 c} \\ &\times |\pi^{1/2}[(1 + \beta_a^2 x^2)D(x) - \sigma(1 + (1 + \beta_a^2)x^2)] - \beta_a^2 x| \\ &\times \left|1 + \beta_a^2 x^2 + \sum_a \frac{\omega_{p,a}^2}{k^2 c^2} x |\pi^{1/2}[(1 + \beta_a^2 x^2)D(x) \right. \\ &\left. - \sigma(1 + (1 + \beta_a^2)x^2)] - \beta_a^2 x\right|^{-2}. \end{aligned} \quad (75)$$

In Figs. 2 and 3, we show the resulting aperiodic parallel electric field and transverse magnetic field fluctuation spectra, respectively, over the whole complex frequency plane, choosing the dimensionless frequency and wavenumber

$$y = \frac{\gamma}{\omega_{p,e}}, \quad \kappa = \frac{ku_e}{\omega_{p,e}}, \quad (76)$$

so that  $x = |y/\kappa|$ . For a pure thermal electron plasma, Eqs. (74) and (75) then read to lowest order in  $\beta_e \ll 1$

$$\frac{2\pi^3 \omega_{p,e}^2 \langle \delta E_{\parallel}^2 \rangle_{\kappa,y}}{k_B T_e} \simeq \frac{1}{|y/\kappa|^2} \frac{|1 - \pi^{1/2} \left| \frac{y}{\kappa} \right| [D\left(\left| \frac{y}{\kappa} \right| \right) - \sigma]|}{\left|1 + \frac{2}{\kappa^2} \left[1 - \pi^{1/2} \left| \frac{y}{\kappa} \right| \left[D\left(\left| \frac{y}{\kappa} \right| \right) - \sigma\right]\right]\right|^2} \quad (77)$$

and

$$\begin{aligned} \frac{2\pi^3 \omega_{p,e}^2 \langle \delta B_{\perp}^2 \rangle_{\kappa,y}}{k_B T_e} &= \frac{2\pi^3 \omega_{p,e}^2 \kappa^2 \langle \delta E_{\perp}^2 \rangle_{\kappa,y}}{\beta_e^2 y^2 k_B T_e} \\ &\simeq \frac{\beta_e^2}{\kappa^3} \left| \pi^{1/2} \left[ \left(1 + \frac{\beta_e^2 y^2}{\kappa^2}\right) D\left(\left| \frac{y}{\kappa} \right| \right) \right. \right. \\ &\quad \left. \left. - \sigma \left(1 + \frac{y^2}{\kappa^2}\right) \right] - \beta_e^2 \left| \frac{y}{\kappa} \right| \right| \\ &\times \left[ 1 + \frac{\beta_e^2 y^2}{\kappa^2} + \frac{\beta_e^2 |y|}{\kappa^3} \right. \\ &\times \left. \left[ \pi^{1/2} \left[ \left(1 + \frac{\beta_e^2 y^2}{\kappa^2}\right) D\left(\left| \frac{y}{\kappa} \right| \right) \right. \right. \right. \\ &\quad \left. \left. \left. - \sigma \left(1 + \frac{y^2}{\kappa^2}\right) \right] - \beta_e^2 \left| \frac{y}{\kappa} \right| \right] \right]^{-2}. \end{aligned} \quad (78)$$

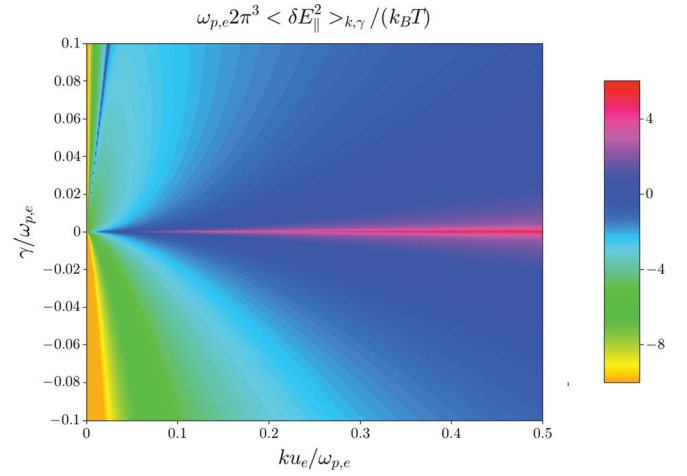


FIG. 2. Colour plot of the spontaneously emitted aperiodic parallel electric field fluctuation spectrum in a thermal nonrelativistic electron-proton distribution in units of  $k_B T_e / (2\pi^3 \omega_{p,e})$  for any nonrelativistic temperature value. Equal electron and proton temperatures ( $T_i = T_e$ ) are adopted. The colour scale is logarithmic in powers of  $e$ .

The right-hand side of Eq. (77) is independent of  $\beta_e^2$ , so that in the units (76), the aperiodic parallel electric field fluctuation spectrum holds for any nonrelativistic plasma temperature. In contrast, the right-hand side of Eq. (78) depends on the value of  $\beta_e$ .

In Fig. 2, we have left out the region  $I=0$ , which deserves special attention, because of the singular dependence  $\propto x^{-1}$  of Eq. (74). Note that we derived the aperiodic approximation holds for real phase speed values  $R \ll |I|$ , setting  $R=0$  for finite  $I$ . However, in the limit  $I \rightarrow 0$ , one better should keep a small but finite value of  $R$ .

## B. Collective, transverse, damped aperiodic mode in the nonrelativistic limit

The bright red ribbon in Fig. 3 clearly indicates that at negative values of  $\gamma < 0$ , a collective transverse purely aperiodic

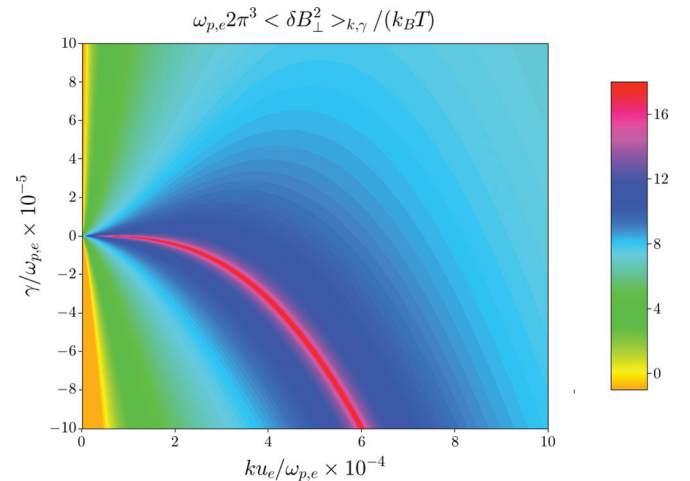


FIG. 3. Contour plot of the spontaneously emitted aperiodic magnetic field fluctuation spectrum in a thermal nonrelativistic electron-proton distribution in units of  $k_B T_e / (2\pi^3 \omega_{p,e})$ . Equal electron and proton temperatures ( $T_i = T_e$ ) and the value  $\beta_e = 10^{-3}$  are adopted. The colour scale is logarithmic in powers of  $e$ .

mode occurs, corresponding according to the third Eq. (1) to a vanishing dispersion function  $\Lambda_T(k, \gamma) = 0$ . With Eq. (58), the dispersion relation of this solely damped mode reads

$$1 + \frac{1}{I^2} + \frac{1}{2} \sum_a \frac{\omega_{p,a}^2 \mu_a}{k^2 c^2} T(I, \mu_a) = 0. \quad (79)$$

With the nonrelativistic approximation (73) and  $\sigma = 2$ , we obtain

$$1 + \frac{1}{I^2} - \sum_a \frac{\omega_{p,a}^2}{k^2 c^2} + \pi^{1/2} \sum_a \frac{\omega_{p,a}^2}{k^2 c^2 \beta_a |I|} \times \left[ (1 + I^2) \left( D\left(\frac{|I|}{\beta_a}\right) - 2 \right) - \frac{2I^2}{\beta_a^2} \right] = 0. \quad (80)$$

For an equal temperature electron-proton plasma and the ratio  $\chi = \sqrt{m_p/m_e} = 43$ , Eq. (80) in terms of the normalized variables (76) reduces to

$$0 = \frac{\beta_e^2 y^2 + \kappa^2}{\beta_e^2 y^2} - \frac{\beta_e^2}{\kappa^2} (1 + \chi^{-2}) + \frac{\pi^{1/2}}{\kappa^3 |y|} \left( -2(1 + \chi)y^2 + (\beta_e^2 y^2 + \kappa^2) \times \left[ D\left(\frac{|y|}{\kappa}\right) + \frac{D\left(\frac{|y|\chi}{\kappa}\right)}{\chi} - 2\left(1 + \frac{1}{\chi}\right) \right] \right). \quad (81)$$

Because of the large value of  $\chi$ , we approximate the dispersion relation (81) by

$$0 = \frac{\beta_e^2 y^2 + \kappa^2}{\beta_e^2 y^2} - \frac{\beta_e^2}{\kappa^2} + \frac{\pi^{1/2}}{\kappa^3 |y|} \left( -2\chi y^2 + (\beta_e^2 y^2 + \kappa^2) \left[ D\left(\frac{|y|}{\kappa}\right) - 2 \right] \right). \quad (82)$$

For all arguments, the function  $D(t) \leq 1$  is smaller than unity. Introducing the phase speed  $x = |y|/\kappa = |\gamma|/ku_e$  then provides the solution

$$\kappa^2 \simeq \frac{\beta_e^2}{1 + \beta_e^2 x^2} [\beta_e^2 x^2 + \pi^{1/2} x (1 + 2\chi x^2)]. \quad (83)$$

To lowest order in  $\beta_e^2 \ll 1$ , Eq. (83) provides the solution

$$x = \frac{1}{\sqrt{6\chi}} \left( \frac{\kappa^2}{\kappa_c^2} \right)^{1/3} \left( \left[ \sqrt{1 + \frac{\kappa_c^4}{\kappa^4}} + 1 \right]^{1/3} - \left[ \sqrt{1 + \frac{\kappa_c^4}{\kappa^4}} - 1 \right]^{1/3} \right), \quad (84)$$

with the characteristic wavenumber

$$\kappa_c = \left( \frac{2\pi}{27\chi} \right)^{1/4} \beta_e = 0.27\beta_e. \quad (85)$$

Consequently, we obtain the solution of the dispersion relation as

$$y = -\kappa x = -6.2 \cdot 10^{-2} \frac{\kappa^{5/3}}{\kappa_c^{2/3}} \left( \left[ \sqrt{1 + \frac{\kappa_c^4}{\kappa^4}} + 1 \right]^{1/3} - \left[ \sqrt{1 + \frac{\kappa_c^4}{\kappa^4}} - 1 \right]^{1/3} \right), \quad (86)$$

with the asymptotics

$$y(\kappa) = - \begin{cases} 4.2 \cdot 10^{-2} \frac{\kappa^3}{\kappa_c^2}, & 0 \leq \kappa \ll \kappa_c \\ 7.8 \cdot 10^{-2} \frac{\kappa^{5/3}}{\kappa_c^{2/3}}, & \kappa_c \ll \kappa, \end{cases} \quad (87)$$

in agreement with the red ribbon in Fig. 3.

## C. Comparison with earlier results

### 1. Parallel electric field fluctuations

In paper I, we were primarily interested in aperiodic thermal fluctuations with positive values of  $\gamma > 0$ . Based on nonrelativistic form factors and plasma dispersion functions, we found in Eq. (67) of paper I for the thermal aperiodic parallel electric field fluctuations

$$\frac{\langle \delta E_{\parallel}^2 \rangle_{k,\gamma>0}^{\text{NR}}}{\sum_a \frac{\omega_{p,a}^2 m_a}{4\pi^3 k^2}} = \gamma \times \frac{1 - \frac{\pi^{1/2} |\gamma|}{|k| u_a} D\left(\frac{\gamma}{ku_a}\right)}{\left| 1 + \sum_a \frac{\omega_{p,a}^2}{2k^2 u_a^2} \left[ 1 - \frac{\pi^{1/2} |\gamma|}{|k| u_a} D\left(\frac{\gamma}{ku_a}\right) \right] \right|^2}. \quad (88)$$

The nonrelativistic temperature limit (74) of the proper relativistic treatment here provides

$$\frac{\langle \delta E_{\parallel}^2 \rangle_{k,\gamma}^{\text{R}}}{\sum_a \frac{\omega_{p,a}^2 m_a}{4\pi^3 k^2}} = \frac{\left| 1 - \pi^{1/2} (1 + \beta_a^2) \frac{\gamma}{ku_a} \left[ D\left(\left| \frac{\gamma}{ku_a} \right| \right) - \sigma \right] \right|}{|\gamma|} \times \left| 1 + 2 \sum_a \frac{\omega_{p,a}^2}{k^2 u_a^2} \left( 1 - \pi^{1/2} (1 + \beta_a^2) \frac{\gamma}{ku_a} \left[ D\left(\left| \frac{\gamma}{ku_a} \right| \right) - \sigma \right] \right) \right|^{-2}. \quad (89)$$

The fluctuation spectrum (89) at positive values of  $\gamma > 0$  differs from the earlier expression (88) of paper I in two aspects:

- (1) it accounts for the missing  $\gamma^2$ -factor noted before,
- (2) it corrects for an erroneous factor 4 in the nonrelativistic longitudinal dispersion function in the denominator.

Apart from these two differences, to lowest order in  $\beta_a \ll 1$ , both expressions agree. These two differences also explain why Fig. 2 looks different from Fig. 3 in paper I at positive values of  $\gamma > 0$ .

However, Eq. (88) should not be used at negative values  $\gamma < 0$ . Here, Eq. (89) gives the correct analytical continuation.

## 2. Perpendicular electric and magnetic field fluctuations

The nonrelativistic form factors and dispersion functions used in paper I provided for the transverse electric and magnetic field fluctuations (Eqs. (68) and (69) in paper I)

$$\langle \delta B_{\perp}^2 \rangle_{k,\gamma}^{\text{NR}} = \frac{k^2 c^2 \langle \delta E_{\perp}^2 \rangle_{k,\gamma}^{\text{NR}}}{\gamma^2} = \sum_a \frac{\omega_{p,a}^2 m_a}{4\pi^3 k^3 c^2} \times \frac{\pi^{1/2} D\left(\frac{\gamma}{ku_a}\right)}{\left|1 + \frac{\gamma^2}{c^2 k^2} + \pi^{1/2} \sum_a \frac{\omega_{p,a}^2}{k^2 c^2} \frac{\gamma}{ku_a} D\left(\frac{\gamma}{ku_a}\right)\right|^2}. \quad (90)$$

The nonrelativistic temperature limit (75) of the proper relativistic treatment yield

$$\langle \delta B_{\perp}^2 \rangle_{k,\gamma}^{\text{R}} = \frac{k^2 c^2 \langle \delta E_{\perp}^2 \rangle_{k,\gamma}^{\text{R}}}{\gamma^2} = \sum_a \frac{\omega_{p,a}^2 m_a \beta_a}{4\pi^3 k^3 c} \times \left| \pi^{1/2} \left[ \left(1 + \frac{\gamma^2}{c^2 k^2}\right) D\left(\frac{\gamma}{ku_a}\right) - \sigma \left(1 + \frac{\gamma^2}{c^2 u_a^2}\right) \right] - \beta_a^2 \left| \frac{\gamma}{ku_a} \right| \times \left| 1 + \frac{\gamma^2}{c^2 k^2} + \sum_a \frac{\omega_{p,a}^2}{k^2 c^2} \left| \frac{\gamma}{ku_a} \right| \times \left( \pi^{1/2} \left[ \left(1 + \frac{\gamma^2}{c^2 k^2}\right) D\left(\frac{\gamma}{ku_a}\right) - \sigma \left(1 + \frac{\gamma^2}{c^2 u_a^2}\right) \right] - \beta_a^2 \left| \frac{\gamma}{ku_a} \right| \right) \right|^2. \quad (91)$$

Apart from a missing  $(\gamma^2/k^2 c^2)$ -factors in the perpendicular form factor, at positive values both fluctuation spectra (90) and (91) agree to lowest order in  $\beta_a \ll 1$ . This agreement can also be seen when comparing Fig. 3 with Fig. 4 in paper I at positive values of  $\gamma > 0$ .

Again, we emphasize that Eq. (90) should not be used at negative values  $\gamma < 0$ . Here, Eq. (91) gives the correct analytical continuation. The correct analytical continuation leads to highest aperiodic magnetic fluctuation levels at negative values of  $\gamma < 0$ , as can be clearly seen by the red structure in Fig. 3.

## D. Ultrarelativistic limit

Here, we start from Eqs. (48) and (49) substituting  $E = (1 - S^2)^{-1/2}$ . With the abbreviations (52), we obtain

$$L(I, \mu_a) = 1 - A_L(I, \mu_a) - \frac{2I^2}{\mu_a^2 K_2(\mu_a)} \int_0^1 dS \frac{e^{-\frac{\mu_a}{\sqrt{1-S^2}}}}{I^2 + S^2} \times \left[ 1 + \frac{\mu_a}{\sqrt{1-S^2}} + \frac{\mu_a^2}{2(1-S^2)} \right] \quad (92)$$

and

$$T(I, \mu_a) = -1 + A_T(I, \mu_a) + \frac{2}{\mu_a^2 K_2(\mu_a)} \int_0^1 dS \frac{e^{-\frac{\mu_a}{\sqrt{1-S^2}}}}{I^2 + S^2} \times \left[ 1 + I^2 - \frac{\mu_a^2}{2} + \frac{\mu_a(1+I^2)}{\sqrt{1-S^2}} + \frac{\mu_a^2(1+I^2)}{2(1-S^2)} \right]. \quad (93)$$

Defining the function

$$P(I, \mu_a) = \int_0^1 dS \frac{e^{-\frac{\mu_a}{\sqrt{1-S^2}}}}{I^2 + S^2}, \quad (94)$$

we can write Eqs. (92) and (93) as

$$L(I, \mu_a) = 1 - A_L(I, \mu_a) - \frac{2I^2}{\mu_a^2 K_2(\mu_a)} \times \left[ P(I, \mu_a) - \mu_a \frac{\partial P(I, \mu_a)}{\partial \mu_a} + \frac{\mu_a^2}{2} \frac{\partial^2 P(I, \mu_a)}{\partial \mu_a^2} \right] \quad (95)$$

and

$$T(I, \mu_a) = -1 + A_T(I, \mu_a) + \frac{2}{\mu_a^2 K_2(\mu_a)} \times \left[ \left( 1 + I^2 - \frac{\mu_a^2}{2} \right) P(I, \mu_a) - \mu_a(1+I^2) \times \frac{\partial P(I, \mu_a)}{\partial \mu_a} + \frac{\mu_a^2(1+I^2)}{2} \frac{\partial^2 P(I, \mu_a)}{\partial \mu_a^2} \right]. \quad (96)$$

For ultrarelativistic temperatures  $\mu_a \ll 1$ , the dominant contributions to the integral (94) result from values  $S \leq \sqrt{1 - \mu_a^2}$ , where we approximate the exponential function  $e^{-\frac{\mu_a}{\sqrt{1-S^2}}} \simeq 1$  by unity. We immediately find the approximation

$$P(I, \mu_a) \simeq \int_0^{\sqrt{1-\mu_a^2}} \frac{dS}{I^2 + S^2} = \frac{1}{I} \arctan \frac{\sqrt{1-\mu_a^2}}{I}, \quad (97)$$

implying

$$\frac{\partial P(I, \mu_a)}{\partial \mu_a} = \frac{-\mu_a}{(I^2 + 1 - \mu_a^2) \sqrt{1 - \mu_a^2}} \quad (98)$$

and

$$\frac{\partial^2 P(I, \mu_a)}{\partial \mu_a^2} = -\frac{I^2 + (1 - \mu_a^2)(1 + 2\mu_a^2)}{(I^2 + 1 - \mu_a^2)^2 (1 - \mu_a^2)^{3/2}}. \quad (99)$$

Collecting terms and using the approximation

$$\frac{1}{K_2(\mu_a \ll 1)} \simeq \frac{\mu_a^2}{2}, \quad (100)$$

we find

$$L(I, \mu_a) \simeq 1 - A_L(I, \mu_a) - I^2 \times \left[ \frac{1}{I} \arctan \frac{\sqrt{1-\mu_a^2}}{I} + \frac{\mu_a^2 [I^2(1-2\mu_a^2) + (1-\mu_a^2)(1-4\mu_a^2)]}{2(I^2 + 1 - \mu_a^2)^2 (1 - \mu_a^2)^{3/2}} \right] \quad (101)$$

and

$$T(I, \mu_a) = -1 + A_T(I, \mu_a) + \frac{1 + I^2 - \mu_a^2}{2I} \arctan \frac{\sqrt{1 - \mu_a^2}}{I} + \frac{\mu_a^2(1 + I^2)[I^2(1 - 2\mu_a^2) + (1 - \mu_a^2)(1 - 4\mu_a^2)]}{2(I^2 + 1 - \mu_a^2)^2(1 - \mu_a^2)^{3/2}}. \quad (102)$$

To lowest order in  $\mu_a \ll 1$ , we obtain for the dispersion functions (101)–(102) in the ultrarelativistic temperature limit

$$L(I, \mu_a) \simeq 1 - I \left[ \arctan \frac{\sqrt{1 - \mu_a^2}}{I} + \frac{\pi\sigma}{2} \right] \quad (103)$$

and

$$T(I, \mu_a) \simeq \frac{1 + I^2}{I} \left[ \arctan \frac{\sqrt{1 - \mu_a^2}}{I} + \frac{\pi\sigma}{2} \right] - 1, \quad (104)$$

which only have a weak dependence on the plasma temperature  $\mu_a$ .

The ultrarelativistic thermal aperiodic fluctuation spectra then are

$$\langle \delta E_{\parallel}^2 \rangle_{k,I} = \sum_a \frac{\omega_{p,a}^2 m_a}{4\pi^3 k^3 c |I|} \times \frac{\left| 1 - I \left[ \arctan \frac{\sqrt{1 - \mu_a^2}}{I} + \frac{\pi\sigma}{2} \right] \right|}{\left| 1 + \sum_a \frac{\mu_a \omega_{p,a}^2}{k^2 c^2} \left( 1 - I \left[ \arctan \frac{\sqrt{1 - \mu_a^2}}{I} + \frac{\pi\sigma}{2} \right] \right) \right|^2} \quad (105)$$

and

$$\langle \delta B_{\perp}^2 \rangle_{k,I} = \frac{\langle \delta E_{\perp}^2 \rangle_{k,\gamma}}{I^2} = \sum_a \frac{\omega_{p,a}^2 m_a}{4\pi^3 k^3 c} \left| (1 + I^2) \left[ \arctan \frac{\sqrt{1 - \mu_a^2}}{I} + \frac{\pi\sigma}{2} \right] - I \right| \times \left| 1 + I^2 + \sum_a \frac{\omega_{p,a}^2 \mu_a I}{2k^2 c^2} \left( (1 + I^2) \times \left[ \arctan \frac{\sqrt{1 - \mu_a^2}}{I} + \frac{\pi\sigma}{2} \right] - I \right) \right|^{-2}, \quad (106)$$

which are shown as colour plots in Figs. 4 and 5, respectively, over the entire complex frequency plane using the dimensionless variables (76). In Fig. 4, we have left out the region  $I = 0$  for reasons explained above. Figs. 4 and 5 indicate that no collective aperiodic modes exist in a thermal ultrarelativistic electron-proton plasma.

## V. SUMMARY AND CONCLUSIONS

We derived the electromagnetic fluctuation spectra in unmagnetized plasmas using fully relativistic dispersion functions and form factors for the important class of isotropic plasma particle distribution functions including in particular relativistic Maxwellian distributions. The general expressions for the electromagnetic fluctuation spectra were

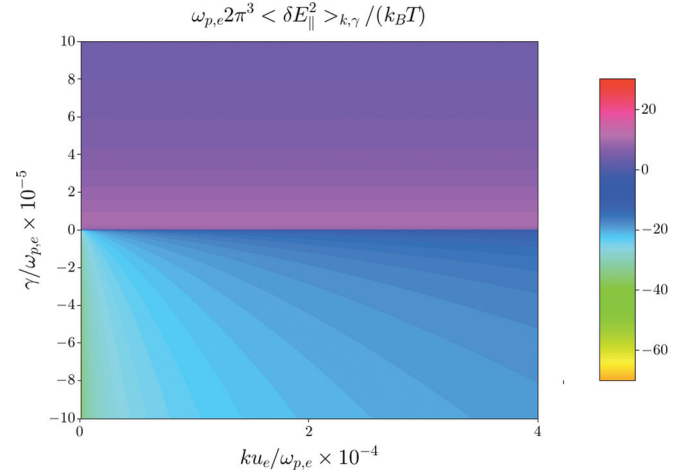


FIG. 4. Colour plot of the spontaneously emitted aperiodic parallel electric field fluctuation spectrum in a thermal ultrarelativistic electron-proton distribution in units of  $k_B T_e / (2\pi^3 \omega_{p,e})$ . Equal electron and proton temperatures ( $T_i = T_e$ ) and  $\beta_e = 10^5$  are adopted. The colour scale is logarithmic in powers of  $e$ .

derived in paper I using the system of the Klimontovich and Maxwell equations for uncorrelated plasma particles, and they are uniquely given in terms of the longitudinal and transverse plasma dispersion functions and the parallel and perpendicular form factors. In order to obtain fluctuation spectra valid in the entire complex frequency plane, the proper analytically continuation to negative imaginary parts of the frequency of the unmagnetized form factors and dispersion functions is essential. The general form of the analytical continuation is derived with no restriction on the real and imaginary parts of the frequency. The derived expressions hold for unmagnetized plasmas of arbitrary composition and arbitrary momentum dependences of the plasma particle distribution functions, and they are covariantly correct within the theory of special relativity.

The results are illustrated for the important special case of isotropic Maxwellian particle distribution functions providing in particular the thermal fluctuations of aperiodic

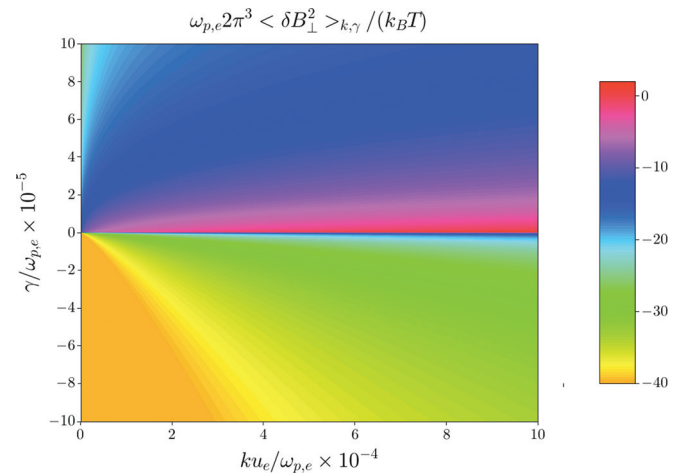


FIG. 5. Colour plot of the spontaneously emitted aperiodic magnetic field fluctuation spectrum in a thermal ultrarelativistic electron-proton distribution in units of  $k_B T_e / (2\pi^3 \omega_{p,e})$ . Equal electron and proton temperatures ( $T_i = T_e$ ) and  $\beta_e = 10^5$  are adopted. The colour scale is logarithmic in powers of  $e$ .

modes, which are of high interest for the formation of cosmic shock waves in collisionless cosmic media and for the origin of cosmological seed magnetic fields in the fully ionized intergalactic medium. In extension to paper I, we made no restriction to the plasma temperature value, deriving for the first time electromagnetic fluctuation spectra in ultrarelativistic thermal plasmas.

In paper I, we were primarily interested in aperiodic fluctuations with positive values of  $\gamma > 0$ . As we demonstrate, the fully relativistic calculations agree favorably well with our earlier results for  $\gamma > 0$  but also provide the correct behavior at negative values of  $\gamma$ , when the nonrelativistic temperature limit is taken. A new collective, transverse, damped aperiodic mode with the damping rate  $\gamma \propto -k^{-5/3}$  is discovered in an isotropic thermal electron-proton plasma with nonrelativistic temperatures. Moreover, we derive the ultrarelativistic limit of the aperiodic thermal fluctuation spectra that are of interest for the unmagnetized fully ionized early Universe plasma and for the interior of hot and compact stars.

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## APPENDIX A: ANALYTICAL CONTINUATIONS

We consider the integral

$$J(E, z) = - \int_{-\sqrt{1-E^{-2}}}^{\sqrt{1-E^{-2}}} \frac{dt}{t-z} = - \int_{-S}^S \frac{dt}{t-R-iI}, \quad (\text{A1})$$

setting  $S = \sqrt{1-E^{-2}}$  and using  $z = R + iI$  according to Eq. (11). For  $I > 0$ , this integral coincides with the integral (26).

To find the analytic continuation of  $J^+$  in the lower half-plane with  $I < 0$ , the limit

$$\lim_{I \rightarrow 0^+} J^+ = \lim_{I \rightarrow 0^-} J^- \quad (\text{A2})$$

has to be fulfilled, where  $J^-$  denotes the integral (A1) for negative values  $I < 0$ . Equation (A1) immediately integrates to the complex logarithmic function

$$J(E, R, I) = \ln X = \ln|X| + i \arg X \quad (\text{A3})$$

where

$$X = \frac{z+S}{z-S} = \frac{R+S+iI}{R-S+iI}. \quad (\text{A4})$$

The complex logarithmic function (A3) is infinitely many-valued. We select the particular argument of  $X$ , which lies in

the interval  $-\pi \leq \arg X \leq \pi$ , as the principal argument and denote this as  $\Phi = \text{Arg } X$ . The resulting principal branch is

$$J_p(E, R, I) = \ln|X| + i\Phi = \frac{1}{2} \ln \frac{(S+R)^2 + I^2}{(S-R)^2 + I^2} - i \left[ \arctan \frac{S-R}{I} + \arctan \frac{S+R}{I} \right], \quad (\text{A5})$$

which is continuous and analytic at all points in the complex  $X$ -plane except the points on the non-positive real axis. This means that the analytic continuation (A2) is fulfilled for all values of  $X$  not lying on the non-positive real axis. However, we have to be careful with the evaluation of the Eq. (A5) in cases where

$$\lim_{I \rightarrow 0^\pm} \Re X \leq 0. \quad (\text{A6})$$

Because the real part of the principal part (A5) is continuous as  $I \rightarrow 0^\pm$ , the requirement (A2) corresponds to

$$\lim_{I \rightarrow 0^+} \Phi^+(R, I) = \lim_{I \rightarrow 0^-} \Phi^-(R, I), \quad (\text{A7})$$

with

$$\Phi(R, I) = - \left[ \arctan \frac{S-R}{I} + \arctan \frac{S+R}{I} \right]. \quad (\text{A8})$$

We first note the symmetry  $\Phi^\pm(-R, I) = \Phi^\pm(R, I)$ , so that it suffices to consider positive values of  $R \geq 0$ . We briefly write  $\Phi^+ = \Phi(R, I > 0)$  and  $\Phi^- = \Phi(R, I < 0)$ .

### 1. Aperiodic fluctuations

For aperiodic fluctuations  $R = 0$ , we find with

$$\Re X = \frac{I^2 - S^2}{S^2 + I^2} \quad (\text{A9})$$

for condition (A7)

$$\lim_{I \rightarrow 0^\pm} \frac{I^2 - S^2}{I^2 + S^2} = -1, \quad (\text{A10})$$

lying on the non-positive real axis in the complex  $X$ -plane for all values of  $S$  or  $E$ . In this case, the principal phase (A8) becomes

$$\Phi(I) = -2 \arctan \frac{S}{I}, \quad (\text{A11})$$

which implies a discontinuity as  $I$  approaches 0 from negative and positive values with

$$\lim_{I \rightarrow 0^+} \Phi^+ = -\pi, \quad \lim_{I \rightarrow 0^-} \Phi^- = +\pi. \quad (\text{A12})$$

Therefore, the requirement (A7) provides

$$\Phi^+ = -2 \arctan \frac{S}{I}, \quad \Phi^- = \Phi^+ - 2\pi, \quad (\text{A13})$$



as the correct analytical continuation. We combine the phase (A13) to the single expression

$$\Phi(I) = -2 \left[ \arctan \frac{S}{I} + \frac{\pi\sigma}{2} \right], \quad (\text{A14})$$

with  $\sigma = 0, 2$  for  $I > 0, I < 0$ , respectively. The phase (A14) holds in the entire complex plane for aperiodic fluctuations. In this case, the principal branch (A5) reduces to

$$J(E, I) = -2i \left[ \arctan \frac{S}{I} + \frac{\pi\sigma}{2} \right]. \quad (\text{A15})$$

## 2. General case $R \neq 0$

For fluctuations with finite  $R$ , the phase (A8) behaves differently for values of  $R \geq S$  and  $R < S$ .

The first case ( $R \geq S$ ) includes superluminal ( $R \geq 1$ ) fluctuations and subluminal ( $R < 1$ ) provided  $E \leq E_c(R)$  with

$$E_c(R) = (1 - R^2)^{-1/2}. \quad (\text{A16})$$

For subluminal fluctuations  $E > E_c(R)$ , the second case  $R < S$  holds. We consider each case in turn.

### 3. Case $R \geq S$

Here, the phase (A8) reads

$$\Phi(R, I) = \arctan \frac{R - S}{I} - \arctan \frac{R + S}{I}, \quad (\text{A17})$$

which fulfils

$$\lim_{I \rightarrow 0^+} \Phi^+ = \lim_{I \rightarrow 0^-} \Phi^- = 0, \quad (\text{A18})$$

and therefore is analytically continuous in the entire complex plane.

### 4. Case $R < S$

Here, the phase (A8)

$$\Phi(R, I) = - \left[ \arctan \frac{S - R}{I} + \arctan \frac{R + S}{I} \right] \quad (\text{A19})$$

exhibits a discontinuity as  $I$  approaches 0 from negative and positive values with

$$\lim_{I \rightarrow 0^+} \Phi^+ = -\pi, \quad \lim_{I \rightarrow 0^-} \Phi^- = +\pi. \quad (\text{A20})$$

Therefore, the requirement (A7) provides

$$\begin{aligned} \Phi^+ &= - \left[ \arctan \frac{S - R}{I} + \arctan \frac{R + S}{I} \right], \\ \Phi^- &= - \left[ \arctan \frac{S - R}{I} + \arctan \frac{R + S}{I} + 2\pi \right], \end{aligned} \quad (\text{A21})$$

which we combine to the single expression

$$\Phi(I) = - \left[ \arctan \frac{S - R}{I} + \arctan \frac{S + R}{I} + \pi\sigma \right], \quad (\text{A22})$$

holding in the entire complex plane. As an aside, we note that Eq. (A22) correctly reproduces Eq. (A14) in the aperiodic limit  $R \rightarrow 0$ .

## 5. Weakly damped/amplified limit

As final case, we consider the weakly damped/amplified limit  $I \rightarrow 0$ . Here, the phase (A17) reduces to

$$\Phi(R \geq 1, I = 0) = \Phi(R < 1, I = 0, E \leq E_c) = 0, \quad (\text{A23})$$

whereas the phase (A22) becomes

$$\Phi(R < 1, I, E > E_c) = -\pi, \quad (\text{A24})$$

so that the principal branch (A5) in this case is given by

$$\begin{aligned} J(R \geq 1, I = 0) &= J(R < 1, I = 0, E < E_c) = \ln \left| \frac{S + R}{S - R} \right|, \\ J(R < 1, I = 0, E \geq E_c) &= \ln \left| \frac{S + R}{S - R} \right| - i\pi. \end{aligned} \quad (\text{A25})$$

## 6. General analytic continuations

All analytical continuations (A14), (A17), (A22)–(A25) can be combined to the single expression

$$\begin{aligned} \Phi(E, R, I) &= - \left( \arctan \frac{S - R}{I} + \arctan \frac{S + R}{I} \right. \\ &\quad \left. + \pi\sigma\Theta[1 - R]\Theta[E - E_c(R)] \right), \end{aligned} \quad (\text{A26})$$

where  $\Theta(x) = (1 + (x/|x|))/2$  denotes the step function and  $\sigma = 0, 1, 2$  for  $I > 0, I = 0$ , and  $I < 0$ , respectively. The corresponding principal branch (A5) is given by

$$\begin{aligned} J(E, R, I) &= \frac{1}{2} \ln \frac{(S + R)^2 + I^2}{(S - R)^2 + I^2} - i \left( \arctan \frac{S - R}{I} \right. \\ &\quad \left. + \arctan \frac{S + R}{I} + \pi\sigma\Theta[1 - R]\Theta[E - E_c(R)] \right), \end{aligned} \quad (\text{A27})$$

holding in the entire complex plane.

## APPENDIX B: PROPERTIES OF THE FUNCTION (70)

In terms of the variable (62), the function (70)

$$G(x, \beta_a) = \frac{D(x) - \frac{\beta_a x}{\sqrt{2}} D\left(\frac{\sqrt{2}}{\beta_a}\right)}{1 - \frac{\beta_a^2 x^2}{2}} \quad (\text{B1})$$

reads

$$G(x, \beta_a) = \frac{4}{\beta_a^2 x^2} e^{\frac{2}{\beta_a^2}} \int_x^\infty dy y e^{-\frac{2y^2}{\beta_a^2}} D(y), \quad (\text{B2})$$

where we substituted  $s = 2y^2/\beta_a^2 x^2$  and used  $I = \beta_a x$  and  $\mu_a = 2/\beta_a^2$  in the integral in Eq. (70). The function  $G(x, \beta_a)$  is monotonically decreasing as a function of  $x$  because its integral representation (B2) provides

$$\frac{\partial G(x, \beta_a)}{\partial x} = -\frac{2}{\beta_a^2 x} [2D(x) + \beta_a^2 G(x, \beta_a)] < 0 \quad (\text{B3})$$

negative values for all values of  $x \in [0, \infty]$ . With  $D(x=0) = 1$ , the maximum value  $G_{\max} = G(0, \beta_a) = 1$  occurs at  $x=0$ .

With the asymptotic expansion<sup>24</sup> for large values of  $x \gg 1$

$$D(x \gg 1) \simeq \frac{1}{\sqrt{\pi x}} \left[ 1 - \frac{1}{2x^2} \right], \quad (\text{B4})$$

both, the integral representation (B2) and Eq. (B1), yield the corresponding asymptotic expansion

$$G(x \gg 1, \beta_a) \simeq \frac{\sqrt{2}}{\beta_a x} D\left(\frac{\sqrt{2}}{\beta_a}\right) \simeq \frac{1}{\sqrt{\pi x}}, \quad (\text{B5})$$

where we approximated

$$D\left(\frac{\sqrt{2}}{\beta_a}\right) \simeq \frac{\beta_a}{\sqrt{2\pi}} \left( 1 - \frac{\beta_a^2}{4} \right). \quad (\text{B6})$$

The function  $G(x, \beta_a)$  approaches the minimum value  $G(\infty, \beta_a) = 0$  at  $x = \infty$ .

Now, we first demonstrate the near equality  $G(x, \beta_a) \simeq D(x)$  for all nonrelativistic velocities  $\beta_a \ll 1$ . Using Eq. (B6), we obtain for Eq. (B1)

$$G(x, \beta_a) \simeq \frac{D(x) - \frac{\beta_a^2 x}{2\sqrt{\pi}} \left( 1 - \frac{\beta_a^2}{4} \right)}{1 - \frac{\beta_a^2 x^2}{2}}. \quad (\text{B7})$$

For small values  $x \leq 1$ , we readily find

$$G(x \leq 1, \beta_a) \simeq D(x) - \frac{\beta_a^2 x}{2\sqrt{\pi}} \geq D(x) - \frac{\beta_a^2}{2\sqrt{\pi}} \simeq D(x). \quad (\text{B8})$$

For large values  $x \geq 1$ , the approximation (B4) yields

$$\begin{aligned} G(x \geq 1, \beta_a) &\simeq \frac{1}{\sqrt{\pi x}} \frac{1 - \frac{1}{2x^2} - \frac{\beta_a^2 x^2}{2} \left( 1 - \frac{\beta_a^2}{4} \right)}{1 - \frac{\beta_a^2 x^2}{2}} \\ &= \frac{1}{\sqrt{\pi x}} \left[ 1 - \frac{1}{2x^2} \left( 1 + \frac{\beta_a^2 x^2}{2} \right) \right] \\ &= D(x) - \frac{\beta_a^2}{4\sqrt{\pi x}}. \end{aligned} \quad (\text{B9})$$

The deviations from  $D(x)$  are at most of the order  $\mathcal{O}(\beta_a^2)$  and therefore negligibly small.

Finally, we calculate the function  $G(x, \beta_a)$  near  $(\sqrt{2}/\beta_a)$  by the Taylor expansion

$$\begin{aligned} D\left(x \simeq \frac{\sqrt{2}}{\beta_a}\right) &\simeq D\left(\frac{\sqrt{2}}{\beta_a}\right) + \left(x - \frac{\sqrt{2}}{\beta_a}\right) \frac{dD}{dx} \Big|_{x=\frac{\sqrt{2}}{\beta_a}} \\ &= D\left(\frac{\sqrt{2}}{\beta_a}\right) \left[ 1 - \frac{4}{\beta_a^2} \left( 1 - \frac{\beta_a x}{\sqrt{2}} \right) \right] + \frac{2\sqrt{2}}{\beta_a \sqrt{\pi}} \left( 1 - \frac{\beta_a x}{\sqrt{2}} \right), \end{aligned} \quad (\text{B10})$$

where we used property (63). Inserting this result into Eq. (B1) provides

$$\begin{aligned} G\left(x \simeq \frac{\sqrt{2}}{\beta_a}, \beta_a\right) &\simeq \frac{\frac{2\sqrt{2}}{\beta_a \sqrt{\pi}} + \left( 1 - \frac{4}{\beta_a^2} \right) D\left(\frac{\sqrt{2}}{\beta_a}\right)}{1 + \frac{\beta_a x}{\sqrt{2}}} \\ &\simeq \frac{\sqrt{\frac{2}{\pi}} \left( 1 - \frac{\beta_a^2}{8} \right) \beta_a}{1 + \frac{\beta_a x}{\sqrt{2}}} \simeq \frac{\sqrt{\frac{2}{\pi}} \beta_a}{1 + \frac{\beta_a x}{\sqrt{2}}}, \end{aligned} \quad (\text{B11})$$

where we used the approximation (B6) in the last step. In particular, we obtain

$$G\left(x = \frac{\sqrt{2}}{\beta_a}, \beta_a\right) = \frac{\beta_a}{\sqrt{2\pi}}, \quad (\text{B12})$$

in full agreement with the approximation (B5).

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